Exercise 41

If $f(x) = x^2/(1+x)$, find $f''(1)$.

Solution

Evaluate the derivative using the quotient rule.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x^2}{1+x}\right) \\
&= \frac{\left[\frac{d}{dx}(x^2)\right](1+x) - \left[\frac{d}{dx}(1+x)\right](x^2)}{(1+x)^2} \\
&= \frac{(2x)(1+x) - (1)(x^2)}{(1+x)^2} \\
&= \frac{x^2+2x}{(1+x)^2}
\end{aligned}$$

Evaluate the second derivative using the quotient rule again.

$$\begin{aligned}
f''(x) &= \frac{d}{dx}[f'(x)] \\
&= \frac{d}{dx}\left[\frac{x^2+2x}{(1+x)^2}\right] \\
&= \frac{\left[\frac{d}{dx}(x^2+2x)\right](1+x)^2 - \left\{\frac{d}{dx}[(1+x)^2]\right\}(x^2+2x)}{(1+x)^4} \\
&= \frac{(2x+2)(1+x)^2 - \left\{\left[\frac{d}{dx}(1+x)\right](1+x) + (1+x)\left[\frac{d}{dx}(1+x)\right]\right\}(x^2+2x)}{(1+x)^4} \\
&= \frac{(2x+2)(1+x)^2 - [(1)(1+x) + (1+x)(1)](x^2+2x)}{(1+x)^4} \\
&= \frac{2(1+x)^3 - 2(1+x)(x^2+2x)}{(1+x)^4} \\
&= \frac{2(1+x)^3 - 2x(1+x)(x+2)}{(1+x)^4} \\
&= \frac{2(1+x)^2 - 2x(x+2)}{(1+x)^3} \\
&= \frac{2}{(1+x)^3}
\end{aligned}$$

Therefore,

$$f''(1) = \frac{2}{(1+1)^3} = \frac{1}{4}.$$